ISRAEL JOURNAL OF MATHEMATICS, Vol. 66, Nos. 1-3, 1989

CONSTRUCTING MODULAR CLASSIFYING SPACES

BY

J. AGUADÉ

Department of Mathematics, Universitat Autònoma de Barcelona, 08193 Bellaterra, Spain

ABSTRACT

Using the homotopy limit construction over a certain small category, we construct spaces whose mod p cohomology algebras are the rings of invariants of some unitary reflection groups of order divisible by p.

1. Introduction

The study of finite loop spaces is related to the classical problem of deciding what polynomial algebras over $\mathbb{F}_p$ can arise as cohomology rings of spaces. The celebrated theorem of Adams and Wilkerson ([1]) solved this problem in the non-modular case, i.e. when the degrees of the generators are prime to p. In this case, the polynomial algebra should be isomorphic, as an algebra over the Steenrod algebra, to the ring of invariants of the mod p reduction of some p-adic reflection group ([1], [9]). Moreover, the irreducible non-modular p-adic reflection groups, as well as the set of primes admisible for each group, are completely classified ([9]). Hence, the interest turns to the modular case, when p divides the degree of some of the generators of the polynomial algebra.

As a first step towards the classification of modular polynomial cohomology algebras and their corresponding homotopy types, one would like to produce, *in a uniform way*, enough examples of spaces of this type. The list of known examples is quite short. First of all, we have the classifying spaces of compact

Received December 21, 1988

connected Lie groups, at appropriate primes, i.e. at primes which do divide the order of the Weyl group but which still produce polynomial cohomology algebras. The next example is Quillen's "p-adic grassmannians" ([18]). Out of these families, only two sporadic examples were known: Two spaces constructed by Zabrodsky ([23]) of ranks 2 and 4, for the primes 3 and 5, respectively. The present work began as an attempt to obtain Zabrodsky's examples in a way which is simple enough to fit in a general pattern which could eventually be used to produce further examples.

According to Dwyer–Miller–Wilkerson ([12], see also [2]) for odd p, a realizable polynomial algebra should be isomorphic to the ring of invariants of a reflection group in $GL_n(\mathbb{F}_p)$ and moreover this group should lift to $GL_n(\hat{\mathbb{Z}}_p)$. It can be also shown ([12]) that the p-adic invariants should be polynomial and so the lifting of G is a complex reflection group and it is a product of groups in the list of Shephard–Todd ([12]). Hence, it is natural to ask about the realizability of the rings of invariants of the modular groups in that list.

The purpose of this paper is to describe a simple, uniform way to construct spaces whose mod p cohomology algebra is the algebra of invariants of any of the groups number 12 (for $p = 3$), 29 (for $p = 5$), 31 (for $p = 5$), 34 (for $p = 7$), 36 (for $p = 5$ and $p = 7$) and 37 (for $p = 7$) in the list of Shephard–Todd ([21]). The invariants of the first group produce the cohomology algebra of Zabrodsky's rank two example. The invariants of group number 31 have the same type as the rank 4 example of Zabrodsky. The groups number 36 and 37 are the Weyl groups of the exceptional Lie groups E_7 and E_8, respectively. As a consequence, we will have new homotopy realizations of the known examples, including some exceptional Lie groups, as well as two more sporadic examples. However, I imagine that the methods in the paper are more interesting than the results themselves and I hope that this work may fit into a future program which handles the classification of finite torsion-free loop spaces.

The main idea in this paper was introduced by Dwyer–Miller–Wilkerson ([12]) and Jackowski–McClure ([15]) who used small diagrams of spaces to obtain homotopy realizations of some classifying spaces of Lie groups. I am indebted to J. Lannes for introducing me to these ideas and for many helpful discussions. In particular, the insight leading to the construction of Zabrodsky's rank two example is due to him and represented the starting point for the present work. I have benefitted from discussions with C. Broto, H. Glover, S. Jackowski, A. Kono, G. Mislin, L. Smith, C. Wilkerson and

Z. Wojtkowiak. I want to thank the École Polythéchnique and the Ohio State University where this work started. I acknowledge also partial support from the CIRIT of the Generalitat de Catalunya. Last, but not least, I express my gratitude to Alex Zabrodsky. I learned from him, in Zürich in 1980, about his two modular loop spaces. His guidance and encouragement was essential, then and thereafter.

2. Diagram categories

In this section we recall some well known facts about homological algebra on a category of diagrams of vector spaces.

Let I be a fixed small category whose objects will be denoted by $0, 1, \dots$. Let $\mathscr{E}$ be the category of $\mathbb{F}_p$-vector spaces, where $\mathbb{F}_p$ is the field with p elements. Denote by $\mathscr{F}$ the functor category whose objects are functors from I to $\mathscr{E}$, i.e. diagrams of vector spaces and linear maps, and whose maps are the natural transformations. Since I is small and $\mathscr{E}$ is abelian, $\mathscr{F}$ is also an abelian category. Moreover, since $\mathscr{E}$ is complete and co-complete with enough injectives and projectives, $\mathscr{F}$ has also enough injective and projective objects. Let us recall how one constructs enough projective objects in $\mathscr{F}$. Let $\bar{I}$ be the category I "made discrete", i.e. the subcategory of I with the same objects but only identity maps. The inclusion functor $f: \bar{I} \to I$ induces a functor $f^*: \mathscr{F} \to \bar{\mathscr{F}}$ where $\bar{\mathscr{F}}$ is the functor category from $\bar{I}$ to $\mathscr{E}$. Notice that an object in $\bar{\mathscr{F}}$ is just a family of vector spaces indexed by I. Since $\mathscr{E}$ admits colimits, f^* has a left adjoint $\tilde{f}$ (the left Kan extension of f). $\tilde{f}$ can be described on the objects in the following explicit way:

$$[\tilde{f}(V_i)_{i \in I}](j) = \bigoplus_{k \in I} \left[\bigoplus_{k \to j} V_k \right].$$

If $\mathbf{M}$ is an object in $\mathscr{F}$, we can consider $\mathbf{K} = \tilde{f} f^* \mathbf{M}$. Then $\mathbf{K}$ projects on $\mathbf{M}$ and, since

$$\mathrm{Hom}_{\mathscr{F}}(\mathbf{K}, \mathbf{L}) = \prod \mathrm{Hom}_{\mathscr{E}}(M_i, L_i),$$

we see that $\mathbf{K}$ is a projective object in $\mathscr{F}$. In a similar way, using the right adjoint of f^*, one can construct enough injective objects in $\mathscr{F}$. Hence, the category $\mathscr{F}$ is suitable for doing homological algebra. In particular, we have functors $\mathrm{Ext}^n_{\mathscr{F}}(-, -)$.

3. The spectral sequence of a homotopy direct limit

Let $\mathscr{S}_*$ be the category of pointed spaces (i.e. pointed simplicial sets). According to Bousfield–Kan ([8]), given a functor $\mathbf{X}$ from I to $\mathscr{S}_*$, we can associate to $\mathbf{X}$ a space $\underrightarrow{\operatorname{holim}}\,\mathbf{X}$ and a spectral sequence

$$E_2^{s,t} \cong \underleftarrow{\lim}^s\{\tilde{H}^{-t}\mathbf{X}\}$$

which is "closely related" to

$$\tilde{H}^*\ \underrightarrow{\operatorname{holim}}\,\mathbf{X}.$$

Here, $\underleftarrow{\lim}^*$ denote the derived functors of the functor $\underleftarrow{\lim}$ on the category $\mathscr{F}$ considered in the section above. These functors can be identified with the functors $\operatorname{Ext}^*_{\mathscr{F}}(\mathbf{F}_p, -)$ where $\mathbf{F}_p$ denotes the constant diagram $\mathbb{F}_p$ (i.e. all objects are $\mathbb{F}_p$ and all maps are the identity).

In particular, if $\mathbf{X}$ is a diagram of connected pointed spaces such that the functors

$$\operatorname{Ext}^i_{\mathscr{F}}(\mathbf{F}_p, \tilde{H}^*\mathbf{X})$$

vanish for $i > 0$, then we have an isomorphism

$$\tilde{H}^*(\underrightarrow{\operatorname{holim}}\,\mathbf{X}) = \underleftarrow{\lim}\,\{\tilde{H}^*\mathbf{X}\}.$$

4. Analysing a certain small category

We are interested in a certain small category I associated to a couple (G, H) where G is a finite group and H is a subgroup of G. We define I as having only two objects 0, 1 and maps:

$$\operatorname{Hom}(0,0) = G$$

$$\operatorname{Hom}(0,1) = \varnothing$$

$$\operatorname{Hom}(1,1) = \{1\}$$

$$\operatorname{Hom}(1,0) = G/H$$

with the obvious composition maps given by the product in G and the left action of G on G/H.

Let $\mathbf{L}$ be the functor from I to $\mathscr{E}$ given by $L_0 = \mathbb{F}_p[G/H]$, $L_1 = \mathbb{F}_p$ with the obvious homomorphisms. Here, $\mathbb{F}_p[G/H]$ denotes the vector space with base G/H. If $\mathbf{M}$ is any object in $\mathscr{F}$ one sees immediately that

$$\mathrm{Hom}_{\mathscr{F}}(\mathbf{L}, \mathbf{M}) \cong M_1$$

and $\mathbf{L}$ is a projective object. The augmentation homomorphism

$$\varepsilon : \mathbb{F}_p[G/H] \to \mathbb{F}_p$$

extends to an epimorphism

$$\mathbf{L} \xrightarrow{\varepsilon} \mathbb{F}_p \to 0.$$

Notice that if $\mathbf{M}$ is an object in $\mathscr{F}$ then M_0 is a G-module.

Let us denote by $\mathbf{K}$ the kernel of ε. We have two exact sequences in $\mathscr{F}$ and the category of $\mathbb{F}_p[G]$-modules, respectively:

$$0 \to \mathbf{K} \to \mathbf{L} \xrightarrow{\varepsilon} \mathbf{F}_p \to 0, \tag{1}$$

$$0 \to K_0 \to \mathbb{F}_p[G/H] \xrightarrow{\varepsilon} \mathbb{F}_p \to 0. \tag{2}$$

Since $\mathbf{L}$ is projective, the functor $\mathrm{Hom}_{\mathscr{F}}(\text{—}, \mathbf{M})$ applied to the exact sequence (1) gives isomorphisms

$$\mathrm{Ext}^i_{\mathscr{F}}(\mathbf{F}_p, \mathbf{M}) \cong \mathrm{Ext}^{i-1}_{\mathscr{F}}(\mathbf{K}, \mathbf{M}), \qquad i \geqq 2.$$

Since $K_1 = 0$, it is easy to see that

$$\mathrm{Ext}^j_{\mathscr{F}}(\mathbf{K}, \mathbf{M}) \cong \mathrm{Ext}^j_G(K_0, M_0), \qquad j \geqq 0,$$

and so we can reduce the computation of some Ext functors in the category $\mathscr{F}$ to ordinary Ext functors in the category of G-modules. Moreover, these functors can be easily related to cohomology of groups in the following way. Let M be a G-module and let K be the kernel of the augmentation homomorphism $\varepsilon : \mathbb{F}_p[G/H] \to \mathbb{F}_p$. We have an exact sequence of vector spaces

$$0 \to M \xrightarrow{\alpha} \mathrm{Hom}_{\varepsilon}(\mathbb{F}_p[G/H], M) \to \mathrm{Hom}_{\varepsilon}(K, M) \to 0. \tag{3}$$

This is also an exact sequence of G-modules with the diagonal action $(g\varphi)(x) = g\varphi(g^{-1}x)$. Since G is a finite group, we have isomorphisms of G-modules

$$\mathrm{Hom}_{\varepsilon}(\mathbb{F}_p[G/H], M) \underset{\Phi}{\xrightarrow{\cong}} \mathrm{Ind}^G_H M \underset{\Psi}{\xrightarrow{\cong}} \mathrm{Coind}^G_H M,$$

$$\Phi(\varphi) = \sum g_i \otimes g_i^{-1}\varphi(g_i),$$

where $g_1, \dots, g_n$ is a set of representatives of G/H. (Notice that, by abuse of notation, we write M instead of $\mathrm{Res}_H^G M$.) Moreover, one can easily check that the composition

$$M \xrightarrow{\alpha} \mathrm{Hom}_{\mathscr{E}}(\mathbb{F}_p[G/H], M) \xrightarrow{\cong} \mathrm{Coind}_H^G M \xrightarrow{\pi} M$$

is the identity isomorphism. Let us consider now the cohomology long exact sequence associated with the exact sequence (3):

$$\cdots \to H^i(G; M) \to H^i(G, \mathrm{Coind}_H^G M) \to H^i(G; \mathrm{Hom}_{\mathscr{E}}(K, M)) \to \cdots .$$

By Shapiro's lemma, the homomorphisms $i: H \to G$, $\pi: \mathrm{Coind}_H^G M \to M$ induce an isomorphism

$$H^*(G; \mathrm{Coind}_H^G M) \cong H^*(H; M).$$

Since

$$H^*(G; \mathrm{Hom}_{\mathscr{E}}(K, M)) \cong \mathrm{Ext}_G^*(K, M),$$

we have

Proposition. *There is a long exact sequence for $i \geqq 1$*

$$\cdots H^i(G; M_0) \xrightarrow{i^*} H^i(H; M_0) \to \mathrm{Ext}_{\mathscr{F}}^{i+1}(\mathbf{F}_p, \mathbf{M}) \to H^{i+1}(G; M_0) \cdots . \quad \blacksquare$$

Hence, the derived functors $\mathrm{Ext}_{\mathscr{F}}^i(\mathbf{F}_p, -)$, $i \geqq 2$, measure the difference between the cohomology of the group G and that of the subgroup H. Finally, we analyse the functor $\mathrm{Ext}_{\mathscr{F}}^1$. If $\mathbf{M}$ is an object in $\mathscr{F}$ we have an exact sequence

$$0 \to \tilde{M}_1 \to M_1 \to \mathrm{Hom}_G(K, M_0) \to \mathrm{Ext}_{\mathscr{F}}^1(\mathbf{F}_p, \mathbf{M}) \to 0$$

where $\tilde{M}_1$ denotes the subspace of M_1 consisting of those vectors $x \in M_1$ such that $\varphi_i(x) = \varphi_j(x)$ for any φ_i, φ_j in $\mathrm{Hom}_I(1, 0)$. On the other side, the analysis above produces a long exact sequence

$$0 \to M_0^G \to M_0^H \to \mathrm{Hom}_G(K, M_0) \to H^1(G; M_0) \to H^1(H; M_0) \cdots$$

where M_0^G denotes, as usual, the subspace of invariant elements. For some objects $\mathbf{M}$, these two exact sequences may be related. For instance, let A be a G-module and denote by $\mathbf{A}$ the object in $\mathscr{F}$ given by $A_0 = A$, $A_1 = A^H$ with the obvious maps, i.e. with the action of G on A and the homomorphisms

$A^H \to A$ obtained by composing the inclusion $A^H \subset A$ with the action of G. In this particular case we have $\tilde{A}_1 = A_0^G$, $A_1 = A_0^H$ and so

$$\mathrm{Ext}^1_{\mathscr{F}}(\mathbf{F}_p, \mathbf{A}) = \mathrm{Ker}\{H^1(G; A) \to H^1(H; A)\}.$$

In this case, Ext^1 measures also the difference between the cohomology of G and that of H.

PROPOSITION. *Let G be a finite group and let H be a subgroup. Assume A is an $\mathbb{F}_p[G]$-module such that the inclusion $\tilde{H} \subset G$ induces $\tilde{H}^*(G; A) \cong \tilde{H}^*(H; A)$. Then, if $\mathbf{A}$ denotes the diagram defined above, we have*

$$\mathrm{Ext}^i_{\mathscr{F}}(\mathbf{F}_p, \mathbf{A}) = 0, \qquad i > 0. \qquad \blacksquare$$

5. The group No. 12

As an abstract group, this is the linear group $GL_2(\mathbb{F}_3)$ of invertible 2×2 matrices over the field with three elements. As a complex reflection group, it is one of the 8 linear liftings of the octahedral group, as a collineation group in (projective) dimension 1 ([21], see also [10]). Shephard–Todd give the following generators for G:

$$S = \frac{1}{\sqrt{2}}\begin{pmatrix} -1 & i \\ -i & 1 \end{pmatrix}, \qquad T = \frac{1}{\sqrt{2}}\begin{pmatrix} \varepsilon & \varepsilon \\ \varepsilon^3 & \varepsilon^7 \end{pmatrix}$$

where ε is a primitive eighth root of unity. Notice that this representation is not 3-adic as given, but it is equivalent to a 3-adic representation such that when we reduce mod 3 we get the natural action of $GL_2(\mathbb{F}_3)$ on $P = \mathbb{F}_3[t_1, t_2]$. The corresponding algebra of invariants is well known since Dickson (see [22]): it is a polynomial algebra over $\mathbb{F}_3$ with generators in degrees 12 and 16.

One can embed Σ_3 into G by representing the transpositions (12) and (23) by the matrices

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \begin{pmatrix} 1 & -1 \\ 0 & -1 \end{pmatrix},$$

respectively, either over $\mathbb{F}_3$ or $\hat{\mathbb{Z}}_3$. The 3-Sylow subgroup of G is the cyclic group of the upper unitriangular matrices and its normalizer N in G is the Borel subgroup of upper triangular matrices and has order $2^2 3$. The center Z of G is of order 2.

6. The groups No. 29 and 31

These groups are closely related to each other, the first being a subgroup of the second one. Let us call them G_{29} and G_{31}, respectively. Shephard gave generating reflections r_1, r_2, r_3, r_4 and r_5 for G_{31} such that r_1, r_2, r_3, r_4 generate the subgroup G_{29} ([19], [20]). These reflections, which are of order two, are given by the matrices

$$r_1 = \frac{1}{2}\begin{bmatrix} 1 & -1 & -1 & -1 \\ -1 & 1 & -1 & -1 \\ -1 & -1 & 1 & -1 \\ -1 & -1 & -1 & 1 \end{bmatrix}, \quad r_2 = \begin{bmatrix} 0 & -i & 0 & 0 \\ i & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix},$$

$$r_3 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad r_4 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix},$$

$$r_5 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Notice that these matrices give directly 5-adic representations of both groups.

Except for the above information on the generating reflections, the study of these two groups is quite involved, due to the fact that the significant references are very old and, in some cases, not easily available. Hence, it seems worthwhile to spend some time collecting information on these groups. Both G_{29} and G_{31} are uniquely determined by their associated collineation groups $\mathcal{G}$ and $\mathcal{G}'$, respectively. $\mathcal{G}'$ is a collineation group of order 11520 in complex projective 3-space. It is generated by homologies of period 2 and was discovered by Klein in 1871 ([16]). It contains 60 homologies whose centers form the vertices of the 15 tetrahedra of a certain projective configuration known as the *Klein configuration*. This group was studied by Bagnera ([4]), Maschke ([17]) and Blichfeldt ([7]) among others. We provide now a description of $\mathcal{G}'$ both as an abstract group and as a collineation group.

Let $H = (\mathbb{Z}/2\mathbb{Z})^6 \rtimes \Sigma_6$ be the group of signed permutations of the set $\{1, \ldots, 6\}$. Let H^+ be the even subgroup of H, i.e. the subgroup of elements of determinant $+1$ in the ordinary representation of dimension 6 of H. Let $\mathcal{H}$

be the quotient of H^+ by its center, which is of order 2. Then, as abstract groups, $\mathscr{G}' \cong \mathscr{H}$.

If V is a complex vector space, we can consider the homomorphism

$$\mathrm{PGL}(V) \xrightarrow{\Phi} \mathrm{PGL}(\Lambda^2 V)$$

given by $\varphi \mapsto \varphi \wedge \varphi$. If $\dim V > 2$, then Φ is a monomorphism. Assume V has dimension 4 with basis e_1, e_2, e_3, e_4. Then $\dim \Lambda^2 V = 6$ and we can consider the following basis:

$$\omega_1 = e_1 \wedge e_2 + e_3 \wedge e_4,$$

$$\omega_2 = i(e_3 \wedge e_4 - e_1 \wedge e_2),$$

$$\omega_3 = e_1 \wedge e_3 + e_4 \wedge e_2,$$

$$\omega_4 = i(e_4 \wedge e_2 - e_1 \wedge e_3),$$

$$\omega_5 = e_1 \wedge e_4 + e_2 \wedge e_3,$$

$$\omega_6 = i(e_2 \wedge e_3 - e_1 \wedge e_4).$$

Let us represent $\mathscr{H}$ in $\mathrm{PGL}(\Lambda^2 V)$ as signed permutations of the basis $\{\omega_i\}$. Then the image of $\mathscr{H}$ is contained in $\Phi(\mathrm{PGL}(V))$ and we have $\mathscr{G}' = \Phi^{-1}(\mathscr{H})$.

$\mathscr{G}$ is a subgroup of index 6 in $\mathscr{G}'$ and is obtained by considering, instead of all permutations of $\{\omega_i\}$, the subgroup of order 120 generated by (23456) and (12)(34)(56).

We will find an embedding of Σ_5 in G_{29} such that the restriction of the representation to Σ_5 is equivalent, over the field $\mathbb{F}_5$, to the canonical representation. By the canonical representation of Σ_n we mean the $(n-1)$-dimensional representation obtained by taking the quotient of the permutation representation in dimension n by the invariant line generated by $(1, \ldots, 1)$. To find this embedding, a computer may be useful. Although the group we are dealing with is quite big, we are only concerned with its 40 reflections, which can be obtained by computing all conjugates of any of the generating reflections r_i. Once we have the matrices of the 40 reflections, we can program a computer to look for a set of four reflections which satisfy the relations of the generating reflections of Σ_5. Once one such set is found, we look for a coordinate system with respect to which the four reflections look like the canonical representation of Σ_5. Notice that it is enough to work mod 5.

Consider the reflections r_1, r_2, r_3 and r_4, where r is the conjugate of r_2 by $r_2r_3r_4r_1$. Consider also the change of basis given by

$$\begin{bmatrix} 0 & 0 & 1 & 3 \\ 3 & 3 & 4 & 1 \\ 4 & 0 & 1 & 0 \\ 0 & 4 & 1 & 0 \end{bmatrix}.$$

Then, in this new basis, the reflections r_1, r_2, r_3 and r_4 are given by

$$r_1 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \qquad r_3 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix},$$

$$r_2 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \qquad r_4 = \begin{bmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & -1 \end{bmatrix}.$$

The center of both G_{29} and G_{31} is cyclic of order 4. The 5-Sylow subgroup of these groups is cyclic of order 5. Using the tables of Benard ([5]) we know that G_{29} has 384 elements of order 5 and G_{31} has 2304 elements of order 5. This allows us to compute the order of the normalizer N of the Sylow subgroup which in both cases is equal to 80.

7. The group No. 34

This group is a bit more accessible than the two groups in the section above, due to several facts. First of all, C. M. Hamill ([13]) published in 1951 an exhaustive study of the configuration formed by the 126 centers of homology of the corresponding collineation group $\mathcal{G}$. Moreover, the group $\mathcal{G}$ is interesting by itself, having a subgroup of index two which is isomorphic to the simple group $PSU_4(\mathbb{F}_3)$ (see the corresponding entry in the atlas of finite groups [11]).

Shephard ([19]) introduced G as the symmetry group of a complex polytope in dimension 6 called $(\frac{1}{3}\gamma_5^3)^{+1}$. G is generated by 6 unitary reflections of period two with respect to the hyperplanes ([19])

$$x_i - x_{i-1} = 0, \quad i = 2, 3, 4, 5;$$

$$x_1 - \omega x_2 = 0,$$

$$x_1 + x_2 + \cdots + x_6 = 0,$$

where ω is a primitive cubic root of unity. (See [19] for the definition of the unitary reflection with respect to an hyperplane.) Notice that, since $\omega \in \hat{\mathbb{Z}}_7$, this is a 7-adic representation.

The center of G is of order 6 and G is the only linear reflection group which projects onto $\mathcal{G}$. The group $\mathcal{G}$ is extensively studied by Hamill in [13]. We can find an embedding of the symmetric group Σ_7 into G in the following way. Let r_i, $i = 1, \ldots, 6$ be the generating reflections as defined above. Hamill determines the set of 126 reflections of G and we see that the reflection r corresponding to the hyperplane $x_6 - \omega x_5$ belongs to G. Then it is easy to check that the reflections r_i, $i \neq 5$ and r generate a subgroup of G isomorphic to Σ_7. Moreover, the change of basis given by the matrix

$$\begin{bmatrix} -4-\omega & 1 & 1 & 1 & 1 & 1 \\ 1 & -4-\omega & 1 & 1 & 1 & 1 \\ 1 & 1 & -4-\omega & 1 & 1 & 1 \\ 1 & 1 & 1 & -4-\omega & 1 & 1 \\ 1 & 1 & 1 & 1 & -4-\omega & 1 \\ \omega & \omega & \omega & \omega & \omega & \omega(-4-\omega) \end{bmatrix}$$

tranforms the above reflections into the canonical representation of Σ_7. Notice also that, by choosing $\omega \equiv 4(7)$, this change of basis is 7-adic.

All elements of order 7 in G are conjugate and Benard ([6]) finds that the size of this conjugacy class is 933120. Then the order of the normalizer N of the 7-Sylow subgroup is $2^2 3^2 7$.

8. The Weyl groups of E_i

Let G_i, $i = 6, 7, 8$ denote the Weyl group of the exceptional Lie group E_i. There is a complete geometrical study of the configurations associated with these groups, done by Hamill in [14]. We see that it is possible to choose a basis $\{y_i\}$ in dimension 9 such that the roots of G_8 lie in the hyperplane $\Sigma y_i = 0$ and they include the roots of the symmetric group Σ_9. The roots of G_7 are obtained by restricting to the hyperplane in $\Sigma y_i = 0$ orthogonal to a root. By choosing appropriate coordinates x_i, $i = 1, \ldots, 8$, we obtain a representation of G_7 in $\Sigma x_i = 0$ whose roots contain the roots of Σ_8. We can proceed further to G_6, but in this case we do not find a symmetric group Σ_7 inside the root system of G_6.

Let ρ be the $(n-1)$-dimensional representation of Σ_n obtained by restricting the permutation representation $\tilde{\rho}$ of Σ_n in dimension n to the hyperplane $\Sigma x_i = 0$. Let ρ^* be the $(n-1)$-dimensional representation of Σ_n obtained by taking the quotient of $\tilde{\rho}$ by the invariant line generated by $(1, \ldots, 1)$ (which we call the canonical representation). It is easy to see that ρ and ρ^* are equivalent over $\mathbb{Z}_{(p)}$ if and only if $n \not\equiv 0(p)$. Since this condition is satisfied by G_7 for $p = 5, 7$ and G_8 for $p = 7$, we obtain that in both cases G_i contains Σ_{i+1} with the canonical representation.

According to Hamill ([14]), G_8 has $2^{12}3^5 5^2$ elements of order 7. Hence, the order of the normalizer of the 7-Sylow subgroup is $2^3 3 7$. The center of G_8 is cyclic of order two. G_7 has $2^9 3^4 5$ elements of order 7 and $2^8 3^3 7$ elements of order 5. The order of the normalizers of the Sylow subgroups for $p = 5$ and $p = 7$ is $2^4 3\ 5$ and $2^2 3\ 7$, respectively. The center of G_8 is cyclic of order two.

9. Lifting a mod p action to the p-adics

Even if we are only interested in the mod p cohomology algebra, we need, in order to perform the construction of the next section, to lift the mod p representation of G to a p-adic representation. Though, in the applications we have in mind, the group G comes with a well-studied p-adic representation, the following lemma, which is a slight generalisation of the Schur–Zassenhaus lemma, will provide us with a lifting of the mod p representation of G to the ring of p-adic integers which will be all right for our purposes.

LEMMA. *Let $H \subset G$ be groups such that*

$$H^i(G; A) \xrightarrow{\cong} H^i(H; A)$$

for all $i > 0$ and all G-modules A whose underlying abelian group is a finite p-group. Let $Q = E/P$ where P is normal in E and is a finite p-group. Given a commutative diagram

$$\begin{array}{ccc} H & \longrightarrow & E \\ {\scriptstyle i}\downarrow & & \downarrow \\ G & \xrightarrow{f} & Q \end{array}$$

there is a homomorphism $\bar{f}: G \to E$ *which makes the diagram commutative.*

PROOF. Let us assume $P = A$ is abelian. The extension $A \lhd E \to Q$ determines an element $\alpha \in H^2(Q; A)$ which maps to zero in $H^2(H; A)$. Hence, $f^*(\alpha) = 0$ and the induced extension $A \lhd E' \to G$ splits. Then, a section $G \to E'$ is given by a derivation $d: G \to A$. We already have a derivation $d: H \to A$ given by a section $s: H \to E'$. Since $H^1(G; A) \cong H^1(H; A)$, d can be extended to G and s can be extended to G. This proves the lemma in the abelian case.

The general case is proved by induction on the order of P. Assume P is not abelian and let $C \subset P$, $C \neq 1$ be the center of P. Consider the extension $P/C \lhd E/C \to Q$. Since $|P/C| < |P|$, the induction hypothesis provides an homomorphism $\hat{f}: G \to E/C$. By applying the lemma in the abelian case to the extension $C \lhd E \to E/C$, we obtain $\bar{f}: G \to E$ with the desired property. ∎

In our case, we will have a commutative diagram

$$\begin{array}{ccc} H & \longrightarrow & GL_n(\hat{\mathbb{Z}}_p) \\ \downarrow & & \downarrow \\ G & \longrightarrow & GL_n(\mathbb{F}_p) \end{array}$$

where H is a subgroup of G. Then, we can extend the mod p representation of G inductively to $GL_n(\mathbb{Z}/p^r\mathbb{Z})$ and to $GL_n(\hat{\mathbb{Z}}_p)$.

10. Realizing rings of invariants

In the preceding sections we have considered some groups G in the list of Shephard–Todd and we have proved that they have some interesting features. In each case, G has a complex representation as a reflection group in dimension n and, moreover, there is a prime p dividing the order of G such that the complex representation of G can be realized over the p-adic integers. G contains a subgroup isomorphic to Σ_{n+1} in such a way that when we restrict the p-adic representation of G to Σ_{n+1} we get the canonical representation on Σ_{n+1} in dimension n, i.e. the quotient of the permutation representation of Σ_{n+1} by its unique eigenvector. The p-Sylow subgroup of G is cyclic of order p and we have computed the order of its normalizer N in G. We can sumarize the results in Table 1.

Denote by P the polynomial algebra $\mathbb{F}_p[t_1, \ldots, t_n]$. We consider P as a graded algebra with an unstable action of the mod p Steenrod algebra through the isomorphism

TABLE 1

Number	Order	Type	p	n	Σ	$\lvert Z\rvert$	$\lvert N\rvert$
12	$2^4\cdot 3$	$\{12, 16\}$	3	2	Σ_3	2	$2^2\cdot 3$
29	$2^9\cdot 3\cdot 5$	$\{8, 16, 24, 40\}$	5	4	Σ_5	4	$2^4\cdot 5$
31	$2^{10}\cdot 3^2\cdot 5$	$\{16, 24, 40, 48\}$	5	4	Σ_5	4	$2^4\cdot 5$
34	$2^9\cdot 3^7\cdot 5\cdot 7$	$\{12, 24, 36, 48, 60, 84\}$	7	6	Σ_7	6	$2^2\cdot 3^2\cdot 7$
36	$2^{10}\cdot 3^4\cdot 5\cdot 7$	$\{4, 12, 16, 20, 24, 28, 36\}$	5	7	Σ_8	2	$2^4\cdot 3\cdot 5$
36	$2^{10}\cdot 3^4\cdot 5\cdot 7$	$\{4, 12, 16, 20, 24, 28, 36\}$	7	7	Σ_8	2	$2^2\cdot 3\cdot 7$
37	$2^{14}\cdot 3^5\cdot 5^2\cdot 7$	$\{4, 16, 24, 28, 36, 40, 48, 60\}$	7	8	Σ_9	2	$2^3\cdot 3\cdot 7$

$$P \cong H^*(BT^n; \mathbb{F}_p),$$

where T^n is an n-dimensional torus.

Let G be any of the groups above. Let H be the subgroup of G generated by Z and Σ, where Z denotes the center of G and Σ is the symmetric group contained in G as described in the preceding sections. Since Σ has no center, we have $H = Z \times \Sigma$. If S denotes a p-Sylow subgroup of Σ, we have $S \subset \Sigma \subset G$ and S is also a p-Sylow subgroup of G because $\lvert G/\Sigma\rvert \not\equiv 0(p)$. Let N' be the normalizer of S in Σ and let $H' = Z \times N' \subset G$. By checking the table above we see that $\lvert N\rvert = \lvert H'\rvert$. This shows that N coincides with the normalizer N_H of S in H. Let A be any $\mathbb{F}_p G$-module. Since S has order p, the cohomology of G is related to the cohomology of S by

$$\tilde{H}^*(G; A) = \tilde{H}^*(S; A)^N.$$

Since $N = N_H$, we deduce

$$(*) \qquad \tilde{H}^*(G; A) = \tilde{H}^*(H; A).$$

Let us take any representation $G \to \mathrm{GL}_n \mathbb{F}_p$ which restricts to the canonical representation of H. In the preceding sections we have seen that such a representation always exists. In this section we will construct, for any of the groups above, a space Z such that

$$H^*(Z; \mathbb{F}_p) \cong P^G$$

as unstable algebras over the Steenrod algebra. The starting point will be a realization of the ring of invariants P^H.

Let **P** denote the diagram associated with the G-module P and the subgroup H of G as in Section 4. Assume that we are able to realize topologically the diagram **P**. This means that we can find a diagram of spaces **X** such that $H^*(\mathbf{X}) = \mathbf{P}$. In other words, we need:

(1) A space X_0 with an action of G such that $H^*(X_0; \mathbb{F}_p) = P$ with the given action of G on P.

(2) A space X_1 such that $H^*(X_1; \mathbb{F}_p) = P^H$.

(3) An H-equivariant map $f: X_0 \to X_1$ which induces the inclusion $P^H \subset P$ in mod p cohomology.

Let Z be the homotopy direct limit of **X**. The Bousfield–Kan spectral sequence of Section 3 relates $H^*(Z; \mathbb{F}_p)$ to the functors

$$\mathrm{Ext}^*_{\mathcal{F}}(\mathbf{F}_p, H^*(\mathbf{X})).$$

The analysis in Section 4 and the main property (*) prove that this spectral sequence collapses to an isomorphism

$$H^*(Z; \mathbb{F}_p) \cong \mathrm{Hom}(\mathbf{F}_p, H^*(\mathbf{X})) \cong P^G.$$

Hence, we are reduced to construct a map $f: X_0 \to X_1$ with the properties (1), (2), and (3) above. Roughly speaking, this will be done by taking $X_0 = BT^n$ and X_1 equal to the quotient of $B\mathrm{SU}(n+1)$ by a suitable Adams map, both completed at p. Since H is the product of a cyclic central group and a symmetric group, this looks all right from a cohomological point of view but it is not accurate enough for our purposes. Notice that we need *true* actions and *true* equivariant maps and not just diagrams which commute up to homotopy. In particular, we need to choose an Adams map which produces an action of $\mathbb{F}_p^*$ on $\mathrm{BSU}(n+1)$ or some equivalent space.

Let q be a prime number different from p and let k be the field obtained by adjoining to $\mathbb{F}_q$ all p^r-th roots of unity, for all r. Let $k_1 \subset k$ be the smallest field containing the p-th roots of unity. By choosing the prime q conveniently, we can suppose that k_1 is an extension of $\mathbb{F}_q$ of degree $p-1$. If we denote by μ_{p^∞} and μ_p the groups of p^r-th roots of unity and p-th roots of unity, respectively, we have a commutative diagram

$$\begin{array}{ccc}
\mathrm{Gal}(k, \mathbb{F}_q) & \longrightarrow & \mathrm{Aut}(\mu_{p^\infty}) \cong \hat{\mathbb{Z}}_p^* \\
\downarrow & & \downarrow \\
\mathrm{Gal}(k_1, \mathbb{F}_q) & \longrightarrow & \mathrm{Aut}(\mu_p) \cong (\mathbb{Z}/p\mathbb{Z})^*
\end{array}$$

where the horizontal arrows are monomorphisms and the vertical ones are epimorphisms. The image of $\mathrm{Gal}(k, \mathbb{F}_q)$ in $\hat{\mathbb{Z}}_p^*$ is a compact subgroup which projects onto $(\mathbb{Z}/p\mathbb{Z})^*$. Then, a straightforward argument shows that this image should contain a $(p-1)$ root of unity and so there is an automorphism α of k such that $\alpha^{p-1}=1$.

The mod p cohomology of the general linear group on a subfield K of $\bar{\mathbb{F}}_q$ was computed by Quillen ([18]). It only depends on the image of

$$\mathrm{Gal}(\bar{\mathbb{F}}_q, K) \to \mathrm{Aut}(\mu_{p^\infty}).$$

For the field k this image is trivial and we obtain $H^*(B\,\mathrm{GL}_{n+1}k; \mathbb{F}_p) \cong H^*(BU(n+1); \mathbb{F}_p)$. This can be extended to $B\mathrm{SL}_{n+1}\,k$. More precisely, we have that the diagonal inclusion $\mu^n_{p^\infty} \subset (k^*)^n \subset \mathrm{SL}_{n+1}\,k$ induces an isomorphism

$$H^*(B\mathrm{SL}_{n+1}\,k; \mathbb{F}_p) \xrightarrow{\cong} H^*(B\mu^n_{p^\infty}; \mathbb{F}_p)^{\Sigma_{n+1}}.$$

We can now define the map $f: X_0 \to X_1$ satisfying the requirements (1), (2) and (3). Recall that we start with a representation of G over $\mathbb{F}_p$ which restricts to the canonical representation of $H = \Sigma \times Z$. The lifting theorem of Section 9 allows us to lift this mod p representation to a representation ρ defined over the p-adic integers. We take $X_0 = B\mu^n_{p^\infty} \times EG$, with G acting on $B\mu^n_{p^\infty}$ through the dual of ρ (we do that in order to get the right action on cohomology). Let $X_1 = B\mathrm{SL}_{n+1}\,k \times_Z EZ$ where Z is the center of G acting on k through an appropriate power of the automorphism α defined above. Finally, the map f is constructed in the following way. Let us denote by $\overline{\mathrm{SL}}_{n+1}\,k$ the subgroup of $\mathrm{GL}_{n+1}\,k$ formed by the matrices with determinant ± 1. It is the kernel of the composition of the determinant homomorphism and the quotient homomorphism $k^* \to k^*/\{\pm 1\}$. Since this homomorphism is a mod p cohomology isomorphism, for p odd, we see that $H^*(\overline{\mathrm{SL}}_{n+1}\,k; \mathbb{F}_p) \cong H^*(\mathrm{SL}_{n+1}\,k; \mathbb{F}_p)$. Then, $\overline{\mathrm{SL}}_{n+1}\,k$ contains the group $\mu^n_{p^\infty} \rtimes \Sigma_{n+1}$ as the subgroup of the linear transformations $x_i \mapsto \lambda_i x_{\sigma(i)}$, $\lambda_i \in \mu_{p^\infty}$, $\sigma \in \Sigma_{n+1}$, $\Pi\lambda_i = 1$. Then f is the following composition:

$$B\mu^n_{p^\infty} \times EG \to B\mu^n_{p^\infty} \times_H EG \to B\mu^n_{p^\infty} \times_H EH \to (B\mu^n_{p^\infty} \times_\Sigma E\Sigma) \times_Z EZ$$

$$\to B(\mu^n_{p^\infty} \rtimes \Sigma) \times_Z EZ \to B\,\overline{\mathrm{SL}}_{n+1}\,k \times_Z EZ.$$

Finally, we take the p-completion of X_0 and X_1 in the sense of Bousfield–Kan. Then conditions (1), (2) and (3) are satisfied and the cohomology of the homotopy limit of the diagram formed by X_0 and X_1 with the action of G on X_0 and the trivial action of H on X_1 is isomorphic to the ring of invariants of G, as desired. We have proved

THEOREM. (1) *For any group G and prime p in Table 1, there is at least one mod p representation of G which restricts to the canonical representation of the subgroup H.*

(2) *For any such representation, there is a space Z such that $H^*(Z; \mathbb{F}_p) \cong P^G$ as algebras over the Steenrod algebra.* ∎

11. Final remark

In this paper we have constructed several spaces whose mod p cohomology algebras are isomorphic, as algebras over the Steenrod algebra, to algebras of invariants of some very particular groups. These groups have been chosen because it seems possible that they play a significant role in an eventual future classification theorem.

There are two points in which the present work is incomplete: First, there is still an infinite family of modular reflection groups with polynomial rings of invariants whose realizability is undecided. It consists of the groups $G(m, n, r)$, $r > 1$ dividing m, m dividing $p - 1$, $p \leqq n$ which appear in number 2 of the list of Shephard–Todd ([21]). On the other hand, as the title of the paper suggests, one would like to construct classifying spaces of new (or old) *finite loop spaces*. To achieve this goal, the cohomology algebras that we realize should be *polynomial algebras*. There are three cases (corresponding to groups No. 29, 31 and 34) in which I have not provided a proof that the mod p ring of invariants is polynomial. I plan to consider these problems in a forthcoming paper ([3]).

REFERENCES

1. J. F. Adams and C. Wilkerson, *Finite H-spaces and algebras over the Steenrod algebra*, Ann. Math. **111** (1980), 95–143.

2. J. Aguadé, *Computing Lannes T functor*, preprint, 1987.

3. J. Aguadé, *Constructing modular classifying spaces II*, in preparation.

4. G. Bagnera, *I gruppi finiti di transformazioni lineari dello spazio che contengono omologie*, Rend. Circ. Mat. Palermo **19** (1905), 1–56.

5. M. Benard, *Schur indices and splitting fields of the unitary reflection groups*, J. Algebra **38** (1976), 318–342.

6. M. Benard, *Characters and Schur indices of the unitary reflection group* $[321]^3$, Pacific J. Math. **58** (1975), 309–321.

7. H. F. Blichfeldt, *The finite discontinuous primitive groups of collineations in four variables*, Math. Ann. **60** (1905), 204–231.

8. A. K. Bousfield and D. M. Kan, *Homotopy Limits, Completions and Localizations*, Lecture Notes in Math. **304**, Springer-Verlag, Berlin, 1972.

9. A. Clark and J. Ewing, *The realization of polynomial algebras as cohomology rings*, Pacific J. Math. **50** (1974), 425–434.

10. A. Cohen, *Finite complex reflection groups*, Ann. Sci. Éc. Norm. Sup. **9** (1976), 379–436.

11. J. H. Conway *et al.*, *Atlas of Finite Groups*, Clarendon Press, Oxford, 1985.

12. W. Dwyer, H. Miller and C. Wilkerson, *Uniqueness of classifying spaces*, mimeographed notes, 1986.

13. C. M. Hamill, *On a finite group of order* 6,531,840, Proc. London Math. Soc. **3** (1953), 401–454.

14. C. M. Hamill, *A collineation group of order* $2^{13}3^{5}5^{2}7$, London Math. Soc. **3** (1953), 54–79.

15. S. Jackowski and J. E. McClure, *Homotopy approximations for classifying spaces of compact Lie groups*, Proc. of the Arcata Conference, 1986, to appear.

16. F. Klein, *Über eine geometrische Repräsentation der Resolventen algebraischer Gleichungen*, Math. Ann. **4** (1871).

17. H. Maschke, *Über die quaternäre, endliche, lineare substitutionsgruppe der Borchardt'schen Moduln*, Math. Ann. **30** (1887), 496–515.

18. D. Quillen, *On the cohomology and K-theory of the general linear group over a finite field*, Ann. Math. **96** (1972), 552–586.

19. G. C. Shephard, *Unitary groups generated by reflections*, Can. J. Math. **5** (1953), 364–383.

20. G. C. Shephard, *Abstract definitions for reflection groups*, Can. J. Math. **9** (1957), 273–276.

21. G. C. Shephard and J. A. Todd, *Finite unitary reflection groups*, Can. J. Math. **6** (1954), 274–304.

22. L. Smith, *Realizability and nonrealizability of Dickson algebras as cohomology rings*, Proc. Am. Math. Soc. **89** (1983), 303–313.

23. A. Zabrodsky, *On the realization of invariant subgroups of* $\pi_* X$, Trans. Am. Math. Soc. **285** (1984), 467–496.